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The Existence of Periodic Surfaces of Some Singularly Perturbed Systems*

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1. INTRODUCTION

Consider a system of first order ordinary differential equations

$$\frac{dx}{dt} = f(x) \tag{1.1}$$

where $x = (x_1, ..., x_n)$ and $f = (f_1, ..., f_n)$. Following Diliberto [1], we define a periodic p-surface, $0 \leqslant p < n$, of the system (1.1) to be a p-surface $X = S(\theta) = S(\theta_1, ..., \theta_p)$ of period ω_i in θ_i, $i = 1, ..., p$, such that a solution of (1.1) starting on it stays on it. Thus, in particular, a periodic solution is a periodic 1-surface and a singular point is a periodic O-surface.

In a recent paper [2], Diliberto established the existence of periodic surfaces to the following two basic types of systems in normal coordinates:

$$\begin{aligned} \frac{d\theta}{dt} &= \mu + \Theta^1(\theta, y, \epsilon)\, y + \Theta^2(\theta, \epsilon) \\ \frac{dy}{dt} &= B(\theta, \epsilon)\, y + a(\theta, \epsilon) + Y(\theta, y, \epsilon)\, y \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \frac{d\theta}{dt} &= \mu + \epsilon[\Theta^1(\theta, y, \epsilon)\, y + \Theta^2(\theta, \epsilon)] \\ \frac{dy}{dt} &= \epsilon[B(\theta, \epsilon)\, y + a(\theta, \epsilon) + Y(\theta, y, \epsilon)\, y] \end{aligned} \tag{1.3}$$

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where θ is a p-vector, y a q-vector and ϵ a real parameter. It was shown in [2] that under suitable conditions the systems (1.2) and (1.3) have unique periodic surfaces $y = S(\theta, \epsilon)$ and $S(\theta, \epsilon) \to 0$ as $\epsilon \to 0$.

In this paper we will consider singularly perturbed systems which generalize (1.2) and (1.3). And the existence, continuous dependence on parameter and uniqueness of periodic surfaces will be established.

2. Fundamental Lemmas

For a vector $x = (x_1, ..., x_n)$ the norm of it is defined by $\| x \| = (\sum_i | x_i |^2)^{1/2}$. The norm of a $n \times n$ matrix A is denoted by $\| A \|$ and $\| A \| = \sup_{\|x\|=1} \| Ax \|$. The following lemmas are well known.

Lemma 2.1. *Let $B(t)$ be a real continuous $n \times n$ matrix and $(B(t)\, x, x) \geqslant b \| x \|^2$, for some $b > 0$ and for all n-vectors x, where $(\cdot,\cdot)$ is the standard inner product. Let $W = W(t, t^0)$ be the fundamental matrix solution of $\dot{x} = B(t)x$. Then*

$$\| W^{-1}(t, t^0)\| \leqslant K e^{-b(t-t^0)}, \quad t > t^0,$$

$$\| W(t, t^0)\| \geqslant K e^{b(t-t^0)}, \quad t > t^0,$$

$$\| W(t, \tau)\| \leqslant K e^{-b(\tau-t)}, \quad t < \tau,$$

where K is a constant.

Corollary 2.1.1. *If in the above lemma the assumption $(B(t)\, x, x) \geqslant b \| x \|^2$ is replaced by $(B(t)\, x, x) \leqslant -b \| x \|^2$, for some $b > 0$, then*

$$\| W^{-1}(t, t^0)\| \leqslant K e^{-b(t^0-t)}, \quad t < t^0,$$

$$\| W(t, t^0)\| \geqslant K e^{b(t^0-t)}, \quad t < t^0,$$

$$\| W(t, \tau)\| \leqslant K e^{-b(t-\tau)}, \quad t > \tau.$$

The next lemma asserts the existence of a unique bounded solution to a linear nonhomogeneous equation and gives an integral formula for it.

Lemma 2.2. *Let $B(t)$ be a real continuous $n \times n$ matrix and $(B(t)\, x, x) \geqslant b \| x \|^2$, for some $b > 0$ and for all x. Then the dieffrential equation*

$$\frac{dx}{dt} = B(t)\, x + h(t), \tag{2.1}$$

where $h(t)$ is bounded, has a unique solution, bounded for all t, given by

$$x(t) = -\int_0^\infty W(t, t+\sigma)\, h(t+\sigma)\, d\sigma$$

where $W(t, t^0)$ is the fundamental matrix solution of

$$\frac{dx}{dt} = B(t)\, x.$$

COROLLARY 2.2.1. *If in the above lemma the assumption $(B(t)\, x, x) \geqslant b \| x \|^2$ is replaced by $(B(t)\, x, x) \leqslant -b \| x \|^2$, for some $b > 0$, then the same conclusion holds, but with the solution now given by*

$$x(t) = \int_{-\infty}^0 W(t, t+\sigma)\, h(t+\sigma)\, d\sigma.$$

The following lemma gives us an integral equation for periodic surfaces, and it will be used later in proving existence.

LEMMA 2.3. *Consider the system of differential equations*

$$\begin{aligned} \frac{d\theta}{dt} &= \Theta(\theta, v, \epsilon) \\ \frac{dv}{dt} &= E(\epsilon)[B(\theta, \epsilon)\, v + a(\theta, \epsilon) + V(\theta, v, \epsilon)\, v], \end{aligned} \tag{2.2}$$

where θ and Θ are p-vectors, v a q-vector, and E, B, V are all $q \times q$ matrices. Let $v = (x, y, z)$, $x = (x_1, \dots, x_{q_1})$, $y = (y_1, \dots, y_{q_2})$, $z = (z_1, \dots, z_{q_3})$ and $q_1 + q_2 + q_3 = q$. Let

$$E(\epsilon) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & \epsilon I_2 & 0 \\ 0 & 0 & \frac{1}{\epsilon} I_3 \end{pmatrix} \quad \textit{and} \quad B(\theta, \epsilon) = \begin{pmatrix} B^1(\theta, \epsilon) & 0 & 0 \\ 0 & B^2(\theta, \epsilon) & 0 \\ 0 & 0 & B^3(\theta, \epsilon) \end{pmatrix},$$

where B^i is a $q_i \times q_i$ matrix and I_i is the $q_i \times q_i$ identity matrix, $i = 1, 2, 3$. Assume that there is $b > 0$ such that $(B^1(\theta, 0)\, x, x) \geqslant b \| x \|^2$, $(B^2(\theta, 0)\, y, y) \geqslant b \| y \|^2$ and $(B^3(\theta, 0)\, z, z) \geqslant b \| z \|^2$. Assume that the R.H.S. of (2.2) has period ω in θ, is Lipschitzian in (θ, v), continuous in (θ, v, ϵ), and that the functions on R.H.S. are all uniformly bounded.

Let $v = S(\theta)$ be a Lipschitzian function of period ω in θ. Let $\theta = \psi(t, t^0, \theta^0, \epsilon; S)$ denote the solution of

$$\frac{d\theta}{dt} = \Theta(\theta, S(\theta), \epsilon), \qquad \theta(t^0) = \theta^0.$$

Let $W(t, t^0, \theta^0, \epsilon; S)$ denote the $q \times q$ matrix solution of

$$\frac{dW}{dt} = E(\epsilon)\, B(\psi(t, t^0, \theta^0, \epsilon; S), \epsilon)\, W, \qquad W(t^0) = I.$$

Then, for small $\epsilon > 0$, if Eq. (2.2) *has a periodic surface it must satisfy the integral equation*

$$S(\theta) = -\int_0^{\infty} W^{-1}(\tau, 0, \theta, \epsilon; S)\, E(\epsilon)[a(\psi, \epsilon) + V(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau,$$

where $\psi = \psi(\tau, 0, \theta, \epsilon; S)$.

This lemma may be proved by applying lemma 2.2 in an obvious way.

COROLLARY 2.3.1. *If in the above lemma the assumptions*

$$(B^1(\theta, 0)\, x, x) \geqslant b \,\|\, x \,\|^2, \qquad (B^2(\theta, 0)\, y, y) \geqslant b \,\|\, y \,\|^2$$

and
$$(B^3(\theta, 0)\, z, z) \geqslant b \,\|\, z \,\|^2$$

are replaced by $(B^1(\theta, 0)\, x, x) \leqslant -b \,\|\, x \,\|^2$, $(B^2(\theta, 0)\, y, y) \leqslant -b \,\|\, y \,\|^2$ and $(B^3(\theta, 0)\, z, z) \leqslant -b \,\|\, z \,\|^2$, for some $b > 0$, then for small $\epsilon > 0$ any periodic surface of the system (2.2) *must satisfy*

$$S(\theta) = \int_{-\infty}^{0} W^{-1}(\tau, 0, \theta, \epsilon; S)\, E(\epsilon)[a(\psi, \epsilon) + V(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau,$$

where $\psi = \psi(\tau, 0, \theta, \epsilon; S)$.

3. BASIC EXISTENCE THEOREMS

For vectors $a = a(\theta)$ and matrices $A = A(\theta)$ whose elements are multiply periodic functions of θ we define the θ-norms by

$$\|\, a \,\|_\theta = \sup_\theta \|\, a(\theta)\|_E \qquad \text{and} \qquad \|\, A \,\|_\theta = \sup_\theta \|\, A(\theta)\|_E$$

where $\|\, a(\theta)\|_E$ and $\|\, A(\theta)\|_E$ are the Euclidean norms as defined in the previous section. The subscripts on the Euclidean and θ-norms will be omitted when the context makes it clear which norm is employed.

THEOREM 3.1. *Consider the system of differential equations*

$$\begin{aligned} \frac{d\theta}{dt} &= \mu + \Theta^1(\theta, z, \epsilon)\, z + \Theta^2(\theta, \epsilon) \\ \frac{dx}{dt} &= B^1(\theta, \epsilon)\, x + a^1(\theta, \epsilon) + X(\theta, z, \epsilon)\, z \\ \epsilon \frac{dy}{dt} &= B^2(\theta, \epsilon)\, y + \epsilon[a^2(\theta, \epsilon) + Y(\theta, z, \epsilon)\, z], \end{aligned} \tag{3.1}$$

where $\theta = (\theta_1, ..., \theta_p)$, $z = (x, y)$, $x = (x_1, ..., x_{q_1})$, $y = (y_1, ..., y_{q_2})$ *and* $q_1 + q_2 = q$. Θ^1, B^1, B^2, X, *and* Y *are matrices with appropriate sizes. The R.H.S. has period* ω *in* θ, *is of class* $C^{(1)}$ *in* (θ, z), *class* $C^{(0)}$ *in* (θ, z, ϵ). *And all functions on R.H.S. are uniformly bounded.*

Let $a^1(\theta, 0) \equiv 0$, $a^2(\theta, 0) \equiv 0$, *and* $X(\theta, 0, \epsilon) \equiv 0$. *Suppose there exist* $b^1 > 0$ *and* $b^2 > 0$ *such that* $(B^1(\theta, 0)\, x, x) \geqslant b^1 \| x \|^2$ *and* $(B^2(\theta, 0)\, y, y) \geqslant b^2 \| y \|^2$. *If* λ *is the largest characteristic root of* $\frac{1}{2}(\Omega + \Omega')$, *where* $\Omega = (\partial\Theta_i^2/\partial\theta_j)$ *at* $\epsilon = 0$ *and* Ω' *is the transpose of* Ω, *we assume that* $\lambda < b^1$ *for all* θ.

Then the system (3.1) *has a unique periodic surface* $z = S(\theta, \epsilon)$ *of class* $C^{(0)}$ *in* (θ, ϵ) *and* $S(\theta, \epsilon) \to 0$ *as* $\epsilon \to 0$.

PROOF. Let us rewrite (3.1) in the form

$$\frac{d\theta}{dt} = \mu + \Theta^1(\theta, z, \epsilon)\, z + \Theta^2(\theta, \epsilon)$$

$$\frac{dz}{dt} = E(\epsilon)[B(\theta, \epsilon)\, z + a(\theta, \epsilon) + Z(\theta, z, \epsilon)\, z],$$

where

$$E(\epsilon) = \begin{pmatrix} I_1 & 0 \\ 0 & \frac{1}{\epsilon} I_2 \end{pmatrix}, \quad B(\theta, \epsilon) = \begin{pmatrix} B^1(\theta, \epsilon) & 0 \\ 0 & B^2(\theta, \epsilon) \end{pmatrix}, \quad a(\theta, \epsilon) = \begin{pmatrix} a^1(\theta, \epsilon) \\ \epsilon a^2(\theta, \epsilon) \end{pmatrix}$$

and

$$Z(\theta, z, \epsilon) = \begin{pmatrix} X(\theta, z, \epsilon) \\ \epsilon Y(\theta, z, \epsilon) \end{pmatrix}.$$

Let $z = (x, y) = (S^1(\theta), S^2(\theta)) = S(\theta)$ be a Lipschitzian function of period ω in θ. Let $\psi(t, t^0, \theta^0, \epsilon; S)$ be the solution of

$$\frac{d\theta}{dt} = \mu + \Theta^1(\theta, S(\theta), \epsilon)\, S(\theta) + \Theta^2(\theta, \epsilon), \qquad \theta(t^0) = \theta^0.$$

Let $W(t, t^0, \theta^0, \epsilon; S)$ be the matrix solution of

$$\frac{dW}{dt} = E(\epsilon)\, B(\psi(t, t^0, \theta^0, \epsilon; S), \epsilon)\, W, \qquad W(t^0) = I.$$

Then, using Lemma 2.3, we see that if for small $\epsilon > 0$ the system (3.1) has a periodic surface it must satisfy

$$S(\theta) = -\int_0^\infty W^{-1}(\tau, 0, \theta, \epsilon; S)\, E(\epsilon)[a(\psi, \epsilon) + Z(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau, \quad (3.2)$$

where $\psi = \psi(\tau, 0, \theta, \epsilon; S)$. Clearly (3.2) is equivalent to the following two integral equations

$$S^1(\theta) = -\int_0^\infty W_1^{-1}(\tau, 0, \theta, \epsilon; S)[a^1(\psi, \epsilon) + X(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau, \tag{3.3}$$

$$S^2(\theta) = -\int_0^\infty W_2^{-1}(\tau, 0, \theta, \epsilon; S)[a^2(\psi, \epsilon) + Y(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau \tag{3.4}$$

where $\psi = \psi(\tau, 0, \theta, \epsilon; S)$ and

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}.$$

Let $\mathscr{S} = \{z = S(\theta) \mid S \text{ is continuous and has period } \omega \text{ in } \theta\}$,

$$\mathscr{S}_{r_0} = \{S \in \mathscr{S} \mid \| S \| < r_0\},$$

$$\mathscr{S}_{r_0, r_1} = \left\{ S \in \mathscr{S}_{r_0} \,\middle|\, \left\| \frac{\partial S}{\partial \theta_j} \right\| < r_1 \quad \text{for} \quad j = 1, \ldots, p \right\}.$$

Let

$$\left\| \frac{\partial S}{\partial \theta} \right\| = \sup_i \left\| \frac{\partial S}{\partial \theta_i} \right\|.$$

Now, using the R.H.S. of (3.2), we define for each small $\epsilon > 0$ a map T on $\mathscr{S}$ by

$$[T(S, \epsilon)](\theta) = -\int_0^\infty W^{-1}(\tau, 0, \theta, \epsilon; S)\, E(\epsilon)[a(\psi, \epsilon) + Z(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau, \tag{3.5}$$

or equivalently we may write $T(S, \epsilon) = (T^1(S, \epsilon), T^2(S, \epsilon))$, where

$$[T^1(S, \epsilon)](\theta) = -\int_0^\infty W_1^{-1}(\tau, 0, \theta, \epsilon; S)[a^1(\psi, \epsilon) + X(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau, \tag{3.6}$$

$$[T^2(S, \epsilon)](\theta) = -\int_0^\infty W_2^{-1}(\tau, 0, \theta, \epsilon; S)[a^2(\psi, \epsilon) + Y(\psi, S(\psi), \epsilon)\, S(\psi)]\, d\tau, \tag{3.7}$$

where $\psi = \psi(\tau, 0, \theta, \epsilon; S)$.

First of all we want to prove that there exists S such that $S = T(S, \epsilon)$. In the following we shall let $\Delta(\rho)$ denote a positive function of the real variable ρ such that $\Delta(\rho) \to 0$ as $\rho \to 0$. To simplify notation we shall let $\Delta(\rho)$ change from line to line. Also, the constant K in each formula below may be different.

From our hypotheses and by Lemma 2.1 we have

$$\| W_1^{-1}(\tau, 0, \theta, \epsilon; S)\| \leqslant K e^{-b^1\tau}, \tag{3.8}$$

$$\| W_2^{-1}(\tau, 0, \theta, \epsilon; S)\| \leqslant K e^{-(1/\epsilon)b^2\tau}. \tag{3.9}$$

The assumptions $a^1(\theta, 0) \equiv 0$, $a^2(\theta, 0) \equiv 0$ and $X(\theta, 0, \epsilon) \equiv 0$ imply that $\| a^1(\psi, \epsilon)\| = \Delta(\epsilon)$, $\| a^2(\psi, \epsilon)\| = \Delta(\epsilon)$ and $\| X(\psi, S, \epsilon)\| = \Delta(\| S \|)$. These facts allow us to estimate $\| T(S, \epsilon)\|$. Using (3.8) and (3.9), we obtain from (3.6) and (3.7) that

$$\| T^1(S, \epsilon)\| \leqslant \Delta(\epsilon) + \Delta(\| S \|) \, \| S \|$$

and

$$\| T^2(S, \epsilon)\| \leqslant \Delta(\epsilon).$$

Thus

$$\| T(S, \epsilon)\| \leqslant \Delta(\epsilon) + \Delta(\| S \|) \, \| S \|. \tag{3.10}$$

It is clear that S continuous in θ implies $T(S, \epsilon)$ continuous in θ. Also, using the fact

$$\psi(\tau, 0, \theta + \omega, \epsilon; S) = \psi(\tau, 0, \theta, \epsilon; S) + \omega,$$

we see that S has period ω in θ implies that $T(S, \epsilon)$ has period ω in θ.

Now let us choose $r_0' > 0$ such that if $\| S \| < r_0'$ then $\Delta(\| S \|) < \frac{1}{2}$. For this r_0' there exists ϵ_0' such that if $0 < \epsilon < \epsilon_0'$ then $\Delta(\epsilon) < \frac{1}{2} r_0'$. With these restrictions it follows from (3.10) that $T(\mathscr{S}_{r_0'}) \subset \mathscr{S}_{r_0'}$.

Differentiating $T^1(S, \epsilon)$ and $T^2(S, \epsilon)$ in (3.6) and (3.7) with respect to θ_i, one can show (see [3]) that

$$\left\| \frac{\partial T^1(S, \epsilon)}{\partial \theta_i} \right\| \leqslant \Delta(\epsilon) + \Delta(\| S \|) + \Delta(\| S \|) \left\| \frac{\partial S}{\partial \theta} \right\|$$

and

$$\left\| \frac{\partial T^2(S, \epsilon)}{\partial \theta_i} \right\| \leqslant \Delta(\epsilon) + \Delta(\| S \|).$$

Hence

$$\left\| \frac{\partial T(S, \epsilon)}{\partial \theta_i} \right\| \leqslant \Delta(\epsilon) + \Delta(\| S \|) + \Delta(\| S \|) \left\| \frac{\partial S}{\partial \theta} \right\|. \tag{3.11}$$

One can find positive r_1, r_0, and ϵ_0 such that by (3.10) and (3.11) and for $0 < \epsilon < \epsilon_0$

$$T(\mathscr{S}_{r_0, r_1}) \subset \mathscr{S}_{r_0, r_1}$$

and so the map T has a fixed point in $\mathscr{S}_{r_0}$. Thus for each ϵ, $0 < \epsilon < \epsilon_0$, there exists $S(\theta, \epsilon)$ in $\mathscr{S}_{r_0}$ such that $S(\theta, \epsilon) = T(S(\theta, \epsilon), \epsilon)$. Then $z = S(\theta, \epsilon)$ is clearly a periodic surface of the system (3.1).

Let S^1, S^2 be elements of $\mathscr{S}_{r_0, r_1}$ and $0 < \epsilon < \epsilon_0$. Then, using (3.6) and (3.7), it can be shown that

$$\| T(S^1, \epsilon) - T(S^2, \epsilon) \| \leqslant \rho \| S^1 - S^2 \|$$

for some $\rho < 1$ and for all $0 < \epsilon < \epsilon_0$ if r_1, r_0 and ϵ_0 have been chosen small enough. Thus the map T on $\mathscr{S}_{r_0, r_1}$ is a contraction which implies uniqueness of the periodic surface.

Now let $S(\theta, \epsilon)$ be the unique periodic surface of the system (3.1) for each ϵ, $0 < \epsilon < \epsilon_0$. When $\epsilon = 0$, $z \equiv 0$ is clearly the unique periodic surface of (3.1), hence we define $S(\theta, 0) \equiv 0$. We claim that $S(\theta, \epsilon)$ is continuous in ϵ for $0 \leqslant \epsilon < \epsilon_0$. By (3.10) we know that $S(\theta, \epsilon) \to 0$ as $\epsilon \to 0$, hence it is sufficient to prove continuity for $0 < \epsilon < \epsilon_0$.

Let ϵ_1 and ϵ_2 be in the interval $0 < \epsilon < \epsilon_0$, and $S_{\epsilon_j}(\theta) = S(\theta, \epsilon_j)$, $j = 1, 2$. Then one can prove that

$$\| S_{\epsilon_1} - S_{\epsilon_2} \| \leqslant \Delta(\epsilon_1 - \epsilon_2),$$

hence $S(\theta, \epsilon)$ is continuous in ϵ uniformly in θ. The proof of this theorem is thus complete.

THEOREM 3.2. *In Theorem* 3.1 *if we replace the assumption* $X(\theta, 0, \epsilon) \equiv 0$ *by* $X(\theta, z, 0) \equiv 0$, *then the same conclusion holds.*

THEOREM 3.3. *If in both Theorem* 3.1 *and* 3.2 *the assumptions* $(B^1(\theta, 0)\, x, x) \geqslant b^1 \| x \|^2$ *and* $(B^2(\theta, 0)\, y, y) \geqslant b^2 \| y \|^2$ *are replaced by* $(B^1(\theta, 0)\, x, x) \leqslant -b^1 \| x \|^2$ *and* $(B^2(\theta, 0)\, y, y) \leqslant -b^2 \| y \|^2$, *for some* $b^1 > 0$ *and* $b^2 > 0$, *we replace the assumption* $\lambda < b^1$ *by* $\lambda_0 > -b^1$ *where* λ_0 *is the smallest characteristic root of* $\frac{1}{2}(\Omega + \Omega')$. *Then the same conclusion holds.*

COROLLARY 3.4. *If in both Theorem* 3.1 *and* 3.2 *the matrix* $B^1(\theta, \epsilon)$ *is block diagonal,* $B^1(\theta, \epsilon) = \operatorname{diag}(B^{11}(\theta, \epsilon), B^{12}(\theta, \epsilon))$, *with*

$$(B^{11}(\theta, 0)\, x^1, x^1) \geqslant b \| x^1 \|^2 \qquad \textit{and} \qquad (B^{12}(\theta, 0)\, x^2, x^2) \leqslant -c \| x^2 \|^2$$

for some $b > 0$ *and* $c > 0$, *we require that all characteristic roots of* $\frac{1}{2}(\Omega + \Omega')$ *lie between* $-c$ *and* b. *Then the same conclusion holds. And note that this corollary holds when* $B^2(\theta, \epsilon)$ *is also such a block diagonal matrix.*

A simple modification of the proof of Theorem 3.1 yields the following results:

THEOREM 3.5. *Consider the system of differential equations*

$$\begin{aligned}
\frac{d\theta}{dt} &= \mu + \epsilon[\Theta^1(\theta, v, \epsilon)\, v + \Theta^2(\theta, \epsilon)] \\
\frac{dx}{dt} &= B^1(\theta, \epsilon)\, x + a^1(\theta, \epsilon) + X(\theta, v, \epsilon)\, v \\
\frac{dy}{dt} &= \epsilon[B^2(\theta, \epsilon)\, y + a^2(\theta, \epsilon) + Y(\theta, v, \epsilon)\, v] \\
\epsilon \frac{dz}{dt} &= B^3(\theta, \epsilon)\, z + \epsilon[a^3(\theta, \epsilon) + Z(\theta, v, \epsilon)\, v],
\end{aligned} \tag{3.12}$$

where $\theta = (\theta_1, ..., \theta_p)$, $v = (x, y, z)$, $x = (x_1, ..., x_{q_1})$, $y = (y_1, ..., y_{q_2})$, $z = (z_1, .., z_{q_3})$ *and* $q_1 + q_2 + q_3 = q$. $\Theta^1, B^1, B^2, B^3, X, Y$, *and* Z *are matrices with appropriate sizes. The R.H.S. has period* ω *in* θ, *is of class* $C^{(1)}$ *in* (θ, v), *class* $C(^0)$ *in* (θ, v, ϵ). *And all functions are uniformly bounded.*

Let $a^1(\theta, 0) \equiv 0$, $a^2(\theta, 0) \equiv 0$, $a^3(\theta, 0) \equiv 0$, $X(\theta, 0, \epsilon) \equiv 0$ *and* $Y(\theta, 0, \epsilon) \equiv 0$. *Suppose there exist positive real numbers* b^1, b^2 *and* b^3 *such that*

$$(B^1(\theta, 0)\, x, x) \geqslant b^1 \| x \|^2, \qquad (B^2(\theta, 0)\, y, y) \geqslant b^2 \| y^2 \|$$

and

$$(B^3(\theta, 0)\, z, z) \geqslant b^3 \| z \|^2.$$

If λ *is the largest characteristic root of* $\frac{1}{2}(\Omega + \Omega')$, *where* $\Omega = (\partial \Theta_i^2 / \partial \theta_j)$ *at* $\epsilon = 0$ *and* Ω' *is the transpose of* Ω, *we assume that* $\lambda < b^2$ *for all* θ.

Then the system (3.12) *has a unique periodic surface* $v = S(\theta, \epsilon)$ *of class* $C^{(0)}$ *in* (θ, ϵ) *and* $S(\theta, \epsilon) \to 0$ *as* $\epsilon \to 0$.

Remark. If the y-equations of the system (3.12) are absent then the theorem holds without any restriction on λ.

THEOREM 3.6. *In Theorem* 3.5 *if we replace the assumptions* $X(\theta, 0, \epsilon) \equiv 0$ *and* $Y(\theta, 0, \epsilon) \equiv 0$ *by* $X(\theta, v, 0) \equiv 0$ *and* $Y(\theta, v, 0) \equiv 0$, *then the same conclusion holds.*

THEOREM 3.7. *If in both Theorem* 3.5 *and* 3.6 *the assumptions* $(B^1(\theta, 0)\, x, x) \geqslant b^1 \| x \|^2$, $(B^2(\theta, 0)\, y, y) \geqslant b^2 \| y \|^2$ *and* $(B^3(\theta, 0)\, z, z) \geqslant b^3 \| z \|^2$ *are replaced by* $(B^1(\theta, 0)\, x, x) \leqslant -b^1 \| x \|^2$, $(B^2(\theta, 0)\, y, y) \leqslant -b^2 \| y \|^2$ *and* $(B^3(\theta, 0)\, z, z) \leqslant -b^3 \| z \|^2$, *for some* $b^1 > 0$, $b^2 > 0$ *and* $b^3 > 0$, *we replace the assumption* $\lambda < b^2$ *by* $\lambda_0 > -b^2$ *where* λ_0 *is the smallest characteristic root of* $\frac{1}{2}(\Omega + \Omega')$. *Then the same conclusion holds.*

COROLLARY 3.8. *If in both Theorem 3.5 and 3.6 the matrix $B^2(\theta, \epsilon)$ is block diagonal, $B^2(\theta, \epsilon) = \mathrm{diag}(B^{21}(\theta, \epsilon), B^{22}(\theta, \epsilon))$, with $(B^{21}(\theta, 0)\, y^1, y^1) \geqslant b \| y^1 \|^2$ and $(B^{22}(\theta, 0)\, y^2, y^2) \leqslant -c \| y^2 \|^2$ for some $b > 0$ and $c > 0$, we require that all characteristic roots of $\frac{1}{2}(\Omega + \Omega')$ lie between $-c$ and b. Then the same conclusion holds. And note that this corollary holds when $B^1(\theta, \epsilon)$ and $B^3(\theta, \epsilon)$ are also such block diagonal matrices.*

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